Reflection invariance of the current in the totally asymmetric simple exclusion process with disorder

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(Received 27 May 1998)

We show that for the totally asymmetric simple exclusion process on a ring, with arbitrary choice of hopping rates across different bonds of the system, the current is independent of the direction of the jumps. [S1063-651X(98)00710-7]

PACS number(s): 05.20.−y, 02.50.Ey

In [1] Tripathy and Barma study, among other models, the asymmetric simple exclusion process [2,3] in a one-dimensional ring geometry, with an arbitrary choice of hopping rate for each bond of the system (this is the disorder of our title). They observe in simulations and numerical solutions of small systems that for the totally asymmetric process, in which jumps are allowed in only one direction, the system current is the same (up to a sign) whether the permitted jumps are to the right or the left, although the steady-state probabilities of individual configurations in these two systems do not seem to be simply related. They remark also that the equality of these two currents follows from simple arguments when the system contains one particle (or one hole) or is half filled.

In this paper we show that this symmetry holds for any number of particles. The essential idea is to couple one of these two processes with the time reversal of the other; the two coupled processes have jumps in the same direction and we can show, using the coupling, that the current in the reversed process is at most that in the unreversed process. Exchanging the two processes yields the opposite inequality, completing the proof.

We remark that this essential idea can be used to prove a similar result for the totally asymmetric simple exclusion process in an open system of L sites in which the $L-1$ bonds of the system are assigned arbitrary hopping rates: The current in the process in which particles enter the system at rate α on the left, traverse the system using these rates, and exit at rate β on the right, is equal (up to a sign) to the current when particles enter the system at rate β on the right and exit at rate α on the left. We will, however, give details only for the ring geometry.

Consider then a periodic system with N particles on a set of sites $\Lambda=\{0,1,\ldots,L-1\}$; a configuration of the system is an element $\eta\in\{0,1\}^\Lambda$ satisfying $\Sigma_{i=1}^{L}\eta(i)=N$, where $\eta(i)=1$ if there is a particle at site i and $\eta(i)=0$ otherwise. A bond b is a pair (b_l,b_r), where either $(b_l,b_r)=(i,i+1)$ for some $i\in\{0,1,\ldots,L-2\}$ or $(b_l,b_r)=(L-1,0)$. With each bond b there is associated a positive number $x(b)$, the rate at which particles attempt to cross that bond. For any configuration η and bond b we write η^b for the configuration obtained from η by interchanging the states at sites b_l and b_r.

We now give a construction of the random processes we will consider. For each bond b we introduce an independent sequence of random times $0<\tau_{b,1}<\tau_{b,2}<\cdots$ defining a Poisson process with rate $x(b)$; we also relabel the set consisting of all of these times for all of the bonds, plus the time 0, in increasing order: $0=\tau_0<\tau_1<\tau_2<\cdots$ (the event that any two of the times $\tau_{b,k}$ coincide has probability zero and can be ignored). The times $\tau_j,\ j\geqslant 1$, are the possible transition times for the process. For given initial configurations ζ,η we define two processes $\rho_t=(\rho_t(i))_{i\in\Lambda}$ and $\lambda_t=(\lambda_t(i))_{i\in\Lambda},\ t\geqslant 0$, taking values in the space of configurations, as follows: (i) $\rho_0=\zeta,\ \lambda_0=\eta$; (ii) ρ_t and λ_t are constant on each time interval $[\tau_{j-1},\tau_j)$; and (iii) if $\tau_j=\tau_{b,k}$ for $b=(b_l,b_r)$ and some k, then

$$\rho_{\tau_j}=\begin{cases}\rho^b_{\tau_{j-1}} & \text{if}\quad \rho_{\tau_{j-1}}(b_l)=1,\quad \rho_{\tau_{j-1}}(b_r)=0,\\ \rho_{\tau_{j-1}} & \text{otherwise,}\end{cases}$$
$$\lambda_{\tau_j}=\begin{cases}\lambda^b_{\tau_{j-1}} & \text{if}\quad \lambda_{\tau_{j-1}}(b_l)=0,\quad \lambda_{\tau_{j-1}}(b_r)=1,\\ \lambda_{\tau_{j-1}} & \text{otherwise.}\end{cases}\tag{1}$$

Note that ρ_t describes particles moving to the right and λ_t particles moving to the left. When necessary we indicate the dependence of these processes on the choice of times and of initial configuration by writing $\rho_t(\tau,\zeta)$ or $\lambda_t(\tau,\eta)$, where τ denotes the family of all times $\tau_{b,k}$.

Now we wish to follow the trajectories of individual particles. To do so it is convenient to define L-periodic extensions $\hat\rho_t,\hat\lambda_t$ of ρ_t,λ_t to the entire integer lattice: $\hat\rho_t(i+L)=\hat\rho_t(i)$ for $i\in\mathbb{Z}$ and $\hat\rho_t(i)=\rho_t(i)$ for $i\in\Lambda$, with a similar definition of $\hat\lambda_t$. We will then also think of the rates $x(b)$ and the exchange times τ_b as defined for all bonds $b=(i,i+1)$, again periodically: $x(i,i+1)=x(i+L,i+L+1)$ and $\tau_{(i,i+1),k}=\tau_{(i+L,i+L+1),k}$ for all $i\in\mathbb{Z}$.

Consider first $\hat\rho_t$. Let us consecutively number the particles in the configuration $\hat\rho_0$, choosing arbitrarily some starting particle, so that $r_m(0)$ is the location of particle $m,\ m\in\mathbb{Z}$. The periodicity of the configuration implies that $r_{m+N}(0)=r_m(0)+L$. We may then follow these particles through time in the obvious way: $r_m(t)$ never decreases and an exchange in $\hat\rho_t$ across the bond $b=(i,i+1)$ (and all its L translates) at time $\tau_j=\tau_{b,k}$ corresponds to an increase $r_m(\tau_j)=r_m(\tau_{j-1})+1$ for all m such that $r_m(\tau_{j-1})$ is an L translate of i. The value of $r_m(t)$ depends on τ and ζ as well as on the choice of starting place for the particle numbering, but the total number of jumps (of a representative set of N particles) between times 0 and T,

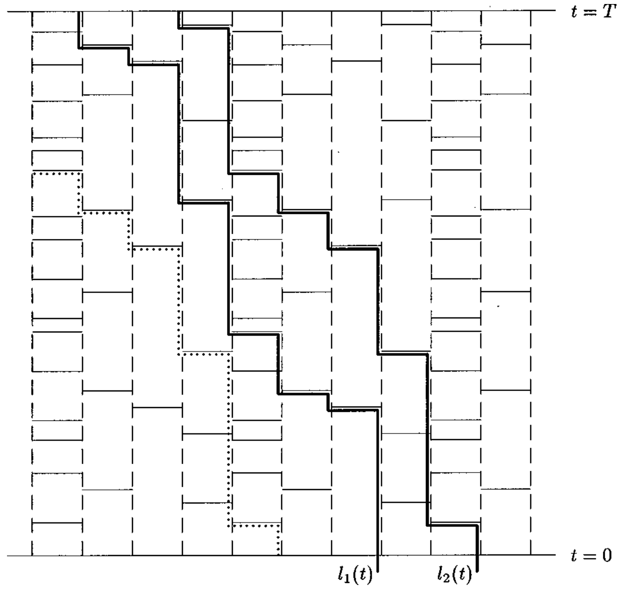

FIG. 1. Space-time diagram of typical trajectories $l_1(t),l_2(t)$ (heavy solid lines) in the case $L=4$, $N=2$. Light dashed vertical lines represent sites; light horizontal lines represent the times $\tau_{b,k}$ at which hopping can occur. Every translate by four sites of one of the trajectories shown is also a trajectory; one of these is shown as a dotted line.

$$J_r(T;\tau,\zeta)=\sum_{m=1}^{N}\,[r_m(T;\tau,\zeta)-r_m(0;\tau,\zeta)], \qquad (2)$$

is independent of this choice. The current j_r for the ρ process is

$$j_r=\frac{1}{LT}E^{\nu_r}J_r(T), \qquad (3)$$

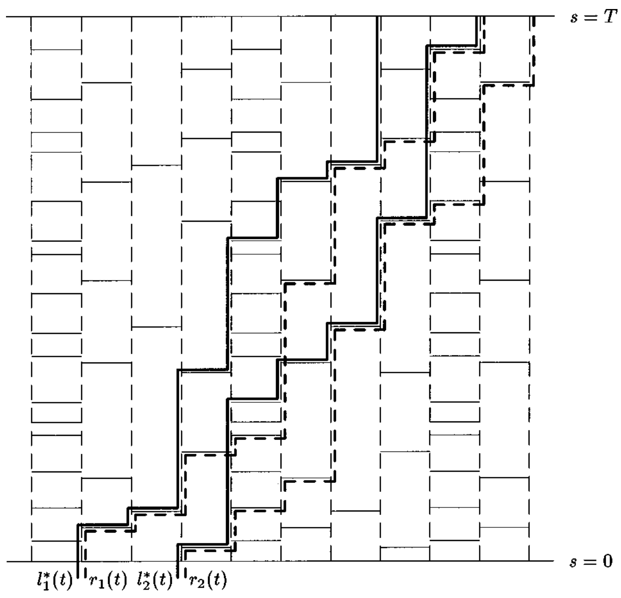

FIG. 2. Typical trajectories $l_1^*(s),l_2^*(s)$ (heavy solid lines) and $r_1(s),r_2(s)$ (heavy dashed lines) in the case $L=4$, $N=2$. The l_i^* trajectories are the reversals of the l_i trajectories of Fig. 1.

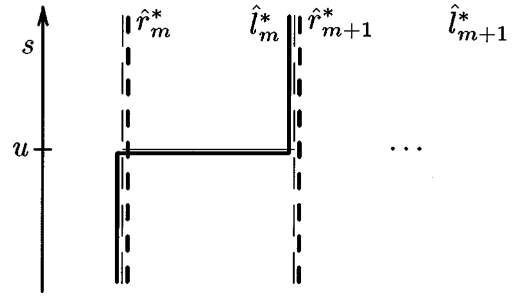

FIG. 3. Behavior of the trajectories $l_m^*(s)$, $l_{m+1}^*(s)$, $r_m(s)$, and $r_{m+1}(s)$ for $s\simeq u$.

where E^{ν_r} denotes expectation with respect to the distribution of the times $\tau_{b,k}$ and the measure ν_r on the initial configuration ζ, and ν_r is the invariant measure for the process ρ_t.

We obtain, similarly, nonincreasing trajectories $l_m(t)=l_m(t;\tau,\eta)$ corresponding to $\hat{\lambda}_t$; $l_m(t)$ is determined by τ, η, and a choice of starting point for the particle numbering. In analogy with Eqs. (2) and (3), the signed total number $J_l(T;\tau,\eta)$ of jumps in the interval $[0,T]$ and the current j_l are given by

$$J_l(T;\tau,\eta)=\sum_{m=1}^{N}\,[l_m(T;\tau,\eta)-l_m(0;\tau,\eta)] \qquad (4)$$

and

$$j_l=\frac{1}{LT}E^{\nu_l}J_l(T), \qquad (5)$$

with ν_l the invariant measure for the process λ_t.

We now want to show that $j_l=-j_r$. To do so we fix a time $T>0$ and let $\hat{\lambda}_s^*$ be the process obtained from $\hat{\lambda}_t$ by time reversal on the interval $[0,T]$:

$$\hat{\lambda}_s^*(\tau,\eta)=\hat{\lambda}_{T-s}(\tau,\eta). \qquad (6)$$

The process $\hat{\lambda}_t(\tau,\eta)$ has trajectories $l_m(t;\tau,\eta)$ obtained as above and the process $\hat{\lambda}_s^*$ has trajectories $l_m^*(s;\tau,\eta)=l_m(T-s;\tau,\eta)$ (the l^* and $\hat{\lambda}_s^*$ processes are left rather than right continuous in time, but this makes no difference in what follows). Suppose that for the bond b there are $K(b)$ times $\tau_{b,k}$ in the interval $[0,T]$, i.e., that $\tau_{b,K(b)}<T<\tau_{b,K(b)+1}$; we then define $\sigma_{b,k}=T-\tau_{b,K(b)+1-k}$. For each b the times $\sigma_{b,k}$ satisfy $0<\sigma_{b,1}<\cdots<\sigma_{b,K(b)}<T$ and have the same distribution as the $\tau_{b,k}$, that is, they form a Poisson process on $[0,T]$; these are the times at which jumps can occur across bond b in the $\hat{\lambda}_s^*$ process.

Consider now simultaneously with $\hat{\lambda}_s^*(\tau,\eta)$ the process $\hat{\rho}_s(\sigma,\zeta)$, with $\sigma=\sigma(\tau)$ as just described and with the initial configuration ζ chosen independently of τ and η. Consider also trajectories $r_m(s;\sigma,\zeta)$ for this process, with the labeling m chosen so that

$$l_m^*(0)\leq r_m(0)<l_m^*(0)+2L \qquad (7)$$

for all m. To achieve this one may, for example, take $r_1(0)$ to be the first particle in the $\hat{\rho}_0$ configuration to the right of

$l_N^*(0)$. A possible space-time picture with two particles in a box of four sites is illustrated in Fig. 1, which shows the unreversed trajectories $l_1(t)$ and $l_2(t)$, and Fig. 2, which shows the time-reversed trajectories $l_1^*(s)$ and $l_2^*(s)$ and the trajectories $r_1(s)$ and $r_2(s)$; for illustrative purposes we have here chosen $l_1^*(0)=r_1(0)$ and $l_2^*(0)=r_2(0)$.

We claim that in fact

$$l_m^*(s)\leqslant r_m(s) \quad \text{for all } m \text{ and all } s\geqslant 0 \tag{8}$$

(see Fig. 2). Granting this claim for the moment, we see, using Eq. (7), that

$$J_r(T;\sigma,\zeta)\geqslant -J_l(T;\tau,\eta)-2LN \tag{9}$$

and hence that

$$j_r=\frac{1}{LT}E^{\nu_r,\nu_l}J_r(T)$$

$$\geqslant -\frac{1}{LT}E^{\nu_r,\nu_l}[J_l(T)+2LN]=-j_l-\frac{2N}{T}. \tag{10}$$

Here E^{ν_r,ν_l} denotes expectation with respect to the Poisson times, the measure ν_l on η, and the measure ν_r on ζ; we have used Eqs. (3) and (5) and the fact that J_r does not depend on η nor J_l on ζ. We emphasize that in Eq. (10), J_r depends on the τ through $\sigma=\sigma(\tau)$ and the first equality follows from Eq. (3) because σ and τ have the same distribution. Since T may be taken arbitrarily large we conclude that $j_r\geqslant -j_l$. However, reversing the argument proves the opposite inequality, so $j_r=-j_l$.

It remains to verify Eq. (8). The key idea is that a particle $r_m(s)$ jumps at the first opportunity, an opportunity provided by the occurrence of an appropriate time $\sigma_{b,k}$ and by an empty site to the particle's right, and that a particle $l_m^*(s)$ at the same site cannot jump any sooner. For a formal proof we argue as follows. Certainly Eq. (8) holds for $s=0$ by the choice of numbering of the particles in $\hat{\rho}_0$ [see Eq. (7)]; if it does not hold for all times, let u be the infimum of the set of times for which it is violated. Then necessarily $u=\sigma_{b,k}$ for some $b=(i,i+1)$ and k. Choose $\epsilon>0$ so that no other transition time σ_j occurs in the interval $[u-\epsilon,u+\epsilon]$. Then Eq. (8) holds at time $u-\epsilon$ and is violated at time $u+\epsilon$, so for some m, $l_m^*(u-\epsilon)=r_m(u-\epsilon)=i$ and $l_m^*(u+\epsilon)=r_m(u+\epsilon)+1=i+1$. However, the fact that l_m^* increases at time u implies that the l_m^* particle is not blocked, so that $l_{m+1}^*(u-\epsilon)>l_m^*(u-\epsilon)+1$, and the fact that r_m does not increase implies that the r_m particle is blocked, so that $r_{m+1}(u-\epsilon)=r_m(u-\epsilon)+1$. The situation must be as shown in Fig. 3. However, then $l_{m+1}^*(u-\epsilon)>r_{m+1}(u-\epsilon)$, contradicting the choice of u.

We thank M. Barma, N. Rajewsky, and G. Tripathy for helpful comments. The work of S.G. was supported by the NSF under Grant No. DMS 95-04556.

[1] Goutam Tripathy and Mustansir Barma, Phys. Rev. E **58**, 1911 (1998).
[2] F. Spitzer, Adv. Math. **5**, 246 (1970).
[3] T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985), and references therein.